# RUNNING TIME ANALYSIS

Problem Solving with Computers-II

```
#include <iostream>
using namespace std;

int main(){
    cout<<"Hola Facebook\n";
    return 0;
}
```

# Problem: Fibonacci Numbers

Definition:

The Fibonacci numbers are the sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,…

Defined by

$F_0 = F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

Problem: Given n, compute $F_n$.

# Which implementation is significantly faster ?

A.

```
F(int n){
   if(n <= 1) return 1
   return F(n−1) + F(n−2)
}
```

B.

```
F(int n){
    Initialize A[0 . . . n]
    A[0] = A[1] = 1

    for i = 2 : n
      A[i] = A[i−1] + A[i−2]

    return A[n]
}
```

C. *Both are almost equally fast*

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```

C. *Both are almost equally fast*

The “right” question is: How does the running time grow?

E.g. How long does it take to compute F(200) recursively?

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How long does it take to compute Fib(200) recursively?
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It will take approximately $2^{92}$ seconds to compute $F_{200}$.

| Time in seconds | Interpretation |
|---|---|
| $2^{10}$ | 17 minutes |
| $2^{20}$ | 12 days |
| $2^{30}$ | 32 years |
| $2^{40}$ | 35000 years<br>(cave paintings) |
| $2^{50}$ | 35 million years ago |
| $2^{70}$ | Big Bang |

**What is the main takeaway so far?**

# How long does it take to compute Fib(200) recursively?

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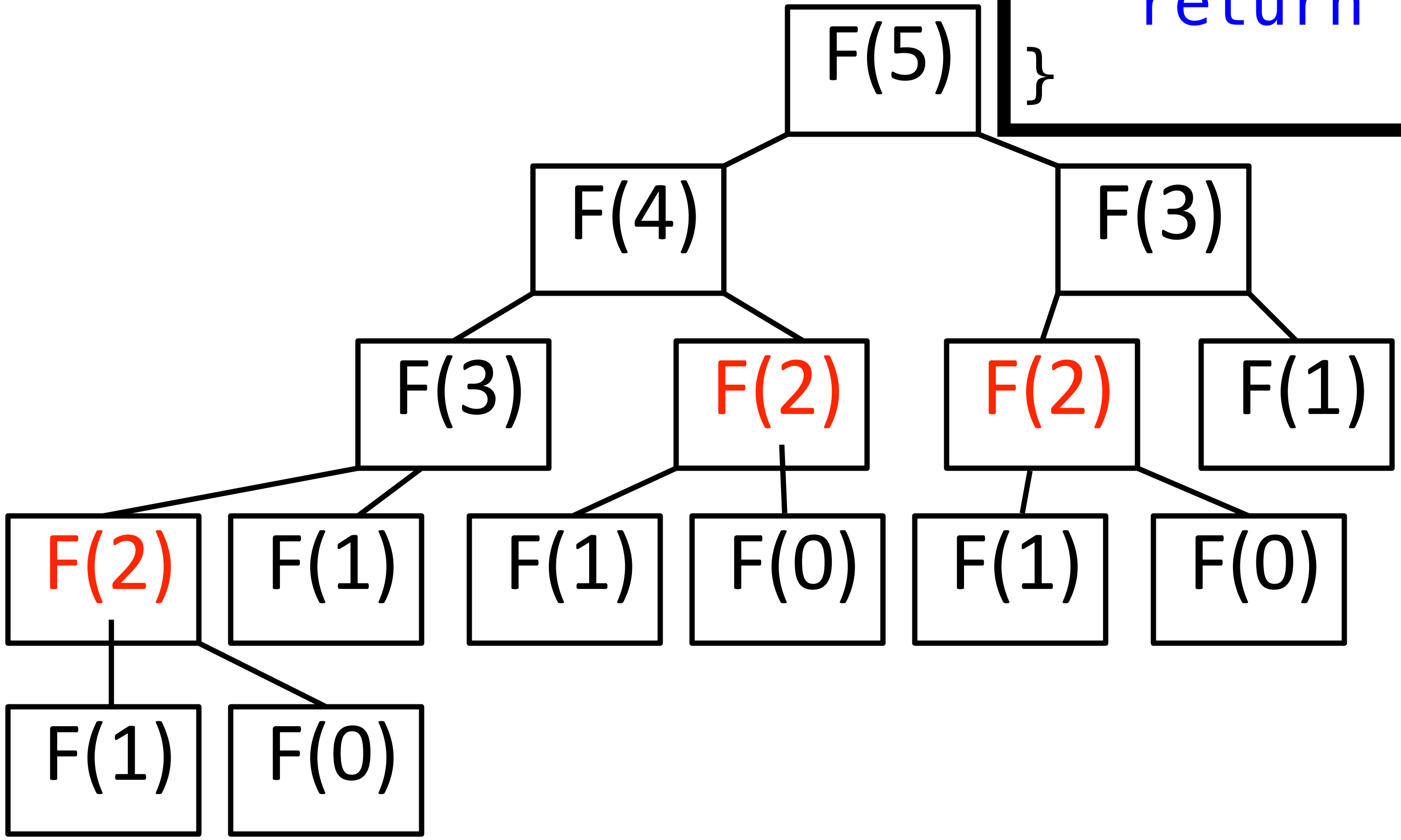
**Questions of interest:**

- Why is Algo A so slow?
- How do we quantify efficiency?
- Is Algo A better than Algo B?
- When will my code finish running?

# Why So Slow?

Too many recursive calls.

```
F(int n){
    if(n <= 1) return 1
    return F(n−1) + F(n−2)
}
```



# Bottom Line

We want to analyze the **impact of the algorithm on running time,** separate from other hardware dependent artifacts that affect time:

- CPU speed
- Memory architecture
- Compiler optimizations
- Background processes

Too much to consider for every analysis **if we analyzed absolute time**

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**Big idea:** Count operations instead of absolute time!

# Machine model used for analysis

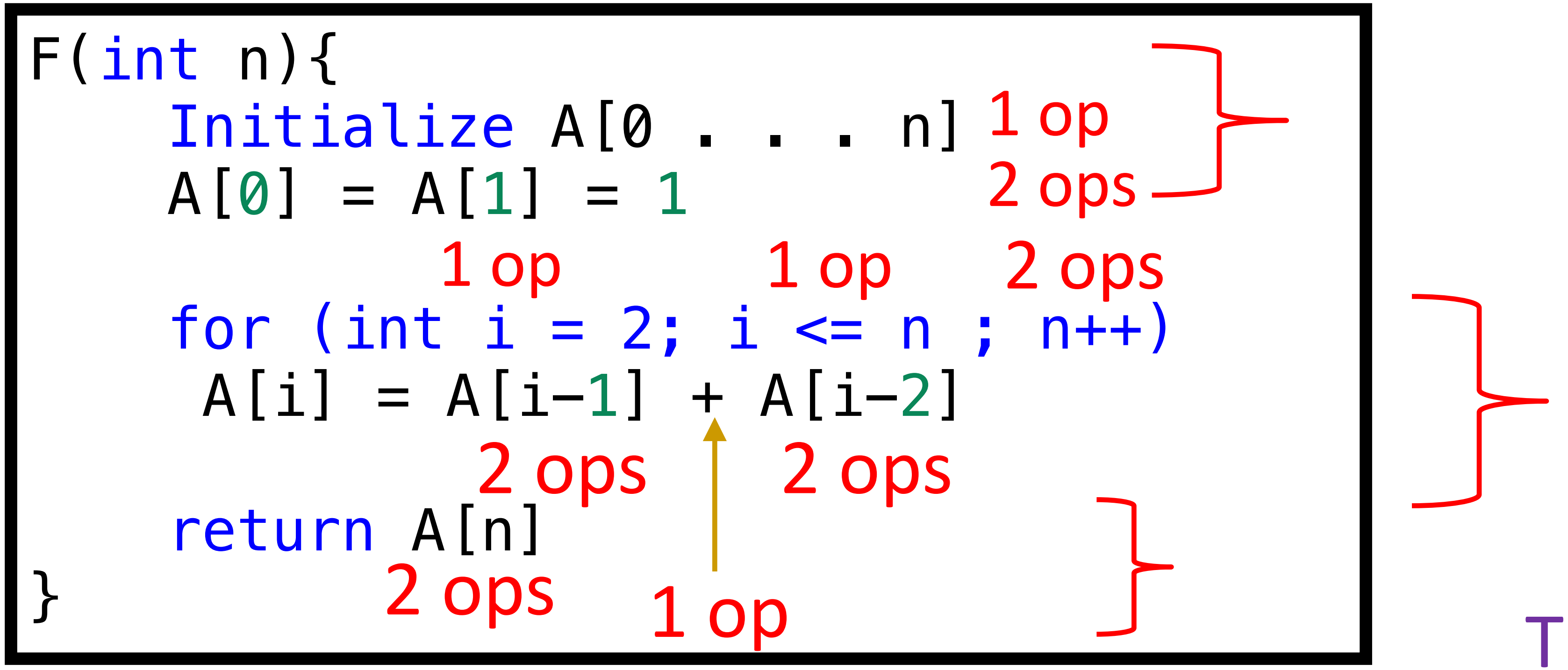
**Big Idea:** Count primitive operations instead of absolute time!

- Every computer can do some **primitive operations** in constant time:
  - Data movement (assignment)
  - Data load/store (accessing an element of an array)
  - Control statements (branch, function call, return)
  - Arithmetic and logical operations

- By inspecting the pseudo-code, we can count the number of primitive operations executed by an algorithm
- **Assumption:** each primitive operation takes a **constant amount of time**

# Iterative Fibonacci Algorithm

Lets compute T(n) = **number of primitive operations** to execute **F**(n)



```
F(int n){
    Initialize A[0 . . . n]
    A[0] = A[1] = 1

    for (int i = 2; i <= n ; n++)
     A[i] = A[i-1] + A[i-2]

    return A[n]
}
```

T(n) =

# Iterative Fibonacci Algorithm

Lets compute T(n) = **number of lines of code** **`F`**(n) needs to execute.

```
F(int n){
    Initialize A[0 . . . n]
    A[0] = A[1] = 1

    for i = 2 : n
     A[i] = A[i−1] + A[i−2]

    return A[n]
}
```

2 lines

2(n-1) lines

1 line

T(n) = 2n+1

# Effect of constant factors

For the iterative fib, we derived two expressions for the running time

$T(n) = 10n - 3$

$T(n) = 2n + 1$

Discuss: how much do the constant factors matter as n gets large?

- Think about $10n - 3$ vs. $10n$ and $2n + 1$ vs. $2n$
- What about $10n$ vs $2n$?

# Analogy: Types of roads and orders of growth

Think of algorithms as **cars traveling a distance.**

- **Running time T(n):** Effort (or fuel) needed to complete the trip
- **Input sizen n**: The distance the car needs to go

10n vs. 2n

**SUV on a highway**

**Sedan on a highway**

Both cars take a similar level of effort (linear) when traveling on a highway.
Think about effort to drive on a smooth highway vs. winding mountain vs. off-road jungle trek

# Orders of growth

**Analogy:** Trips that need a similar effort have the same **order of growth**

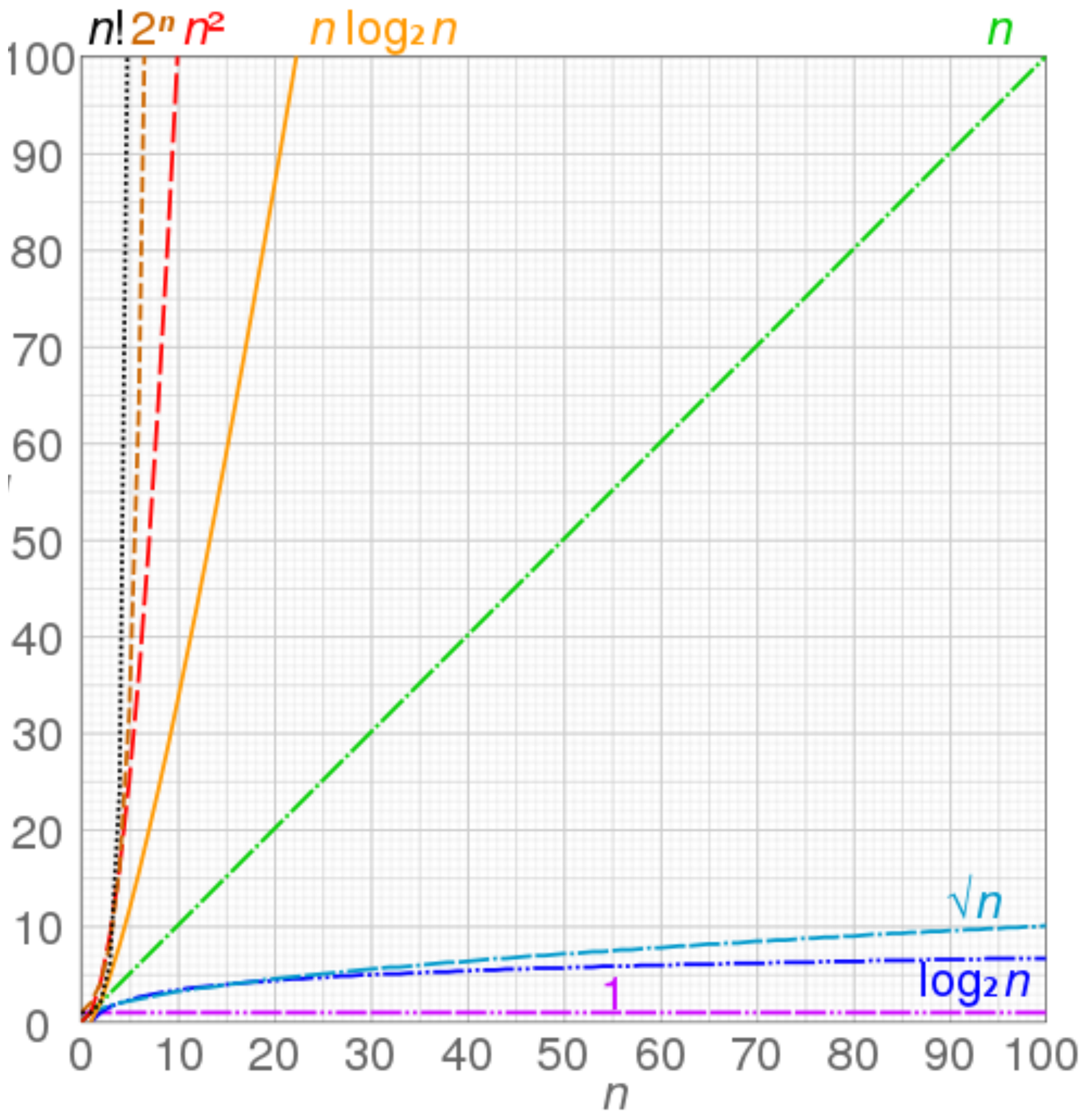
An **order of growth** is a set of functions whose (asymptotic) growth behavior is considered equivalent.

For example, 2*n*, 100*n* and *n* belong to the same order of growth (linear)

Which of the following functions has a higher order of growth?

A. 50n

B. $2n^2$



# Big-O notation

- Big-O notation provides an asymptotic upper bound on the running time
- Its like saying “No matter how bad it gets, the effort won’t exceed this level of difficulty”

# Definition of Big-O

f(n) and g(n) map positive integer inputs to positive reals.

We say f = O(g) if there is a constant $c > 0$ and $k > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n >= k$.

f = O(g)
means that “f grows no faster than g”

# Express in Big-O notation

1. 10000000
2. 3n
3. 6n-2
4. 15n + 44
5. 50nlog(n)
6. $n^2$
7. $n^2$-6n+9
8. $3n^2$+4*log(n)+1000
9. $3^n$ + $n^3$ +log(3*n)

Common sense rules

1. Multiplicative constants can be omitted: $14n^2$ becomes $n^2$ .
2. $n^a$ dominates $n^b$ if a > b: for instance, $n^2$ dominates n.
3. Any exponential dominates any polynomial: $3^n$ dominates $n^5$ (it even dominates $2^n$ ).

**For polynomials, use only leading term, ignore coefficients: linear, quadratic**

# What is the Big O running time of sum()?

```
/* n is the length of the array*/
int sum(int arr[], int n)
{
     int result = 0;
     for(int i = 0; i < n; i+=2)
           result+=arr[i];
     return result;
}
```

A. $O(n^2)$
B. $O(n)$
C. $O(n/2)$
D. $O(\log n)$
E. None of the above

# What is the Big O running time of sum()?

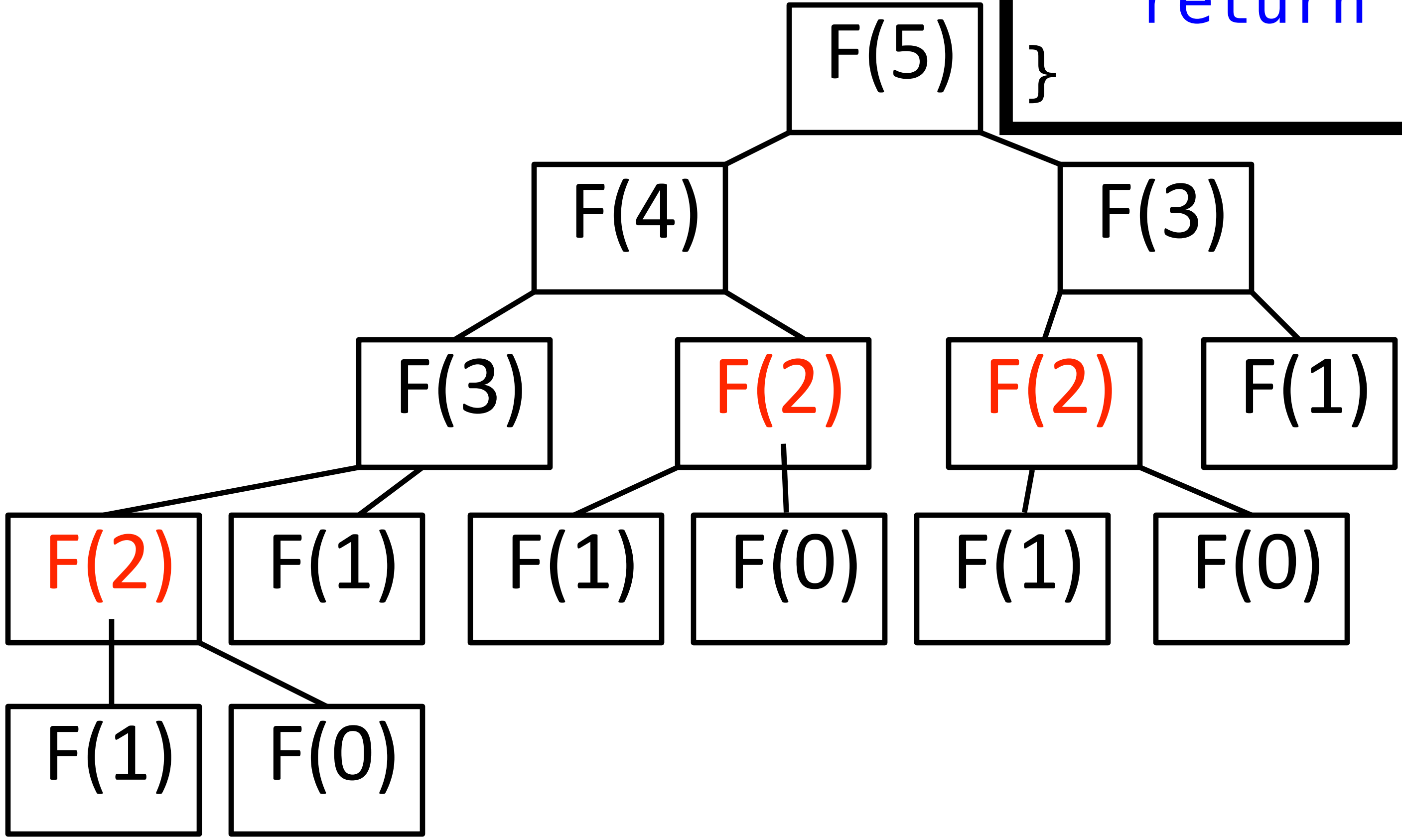
```
/* n is the length of the array*/
int sum(int arr[], int n)
{
      int result = 0;
      for(int i = 1; i < n; i*=2)
              result+=2*arr[i];
       return result;
}
```

A. O($n^2$)
B. O(n)
C. O(n^3)
D. O(log n)
E. None of the above

# Why Big-O is useful in analysis of recursive fib?

Derive $T(n) = O(2^n)$

```
F(int n){
    if(n <= 1) return 1
    return F(n-1) + F(n-2)
}
```



# Emprical Analysis: Recursive Fibonacci Running Time

For recursive fibonacci algorithm, we derived that $T(n) = O(2^n)$

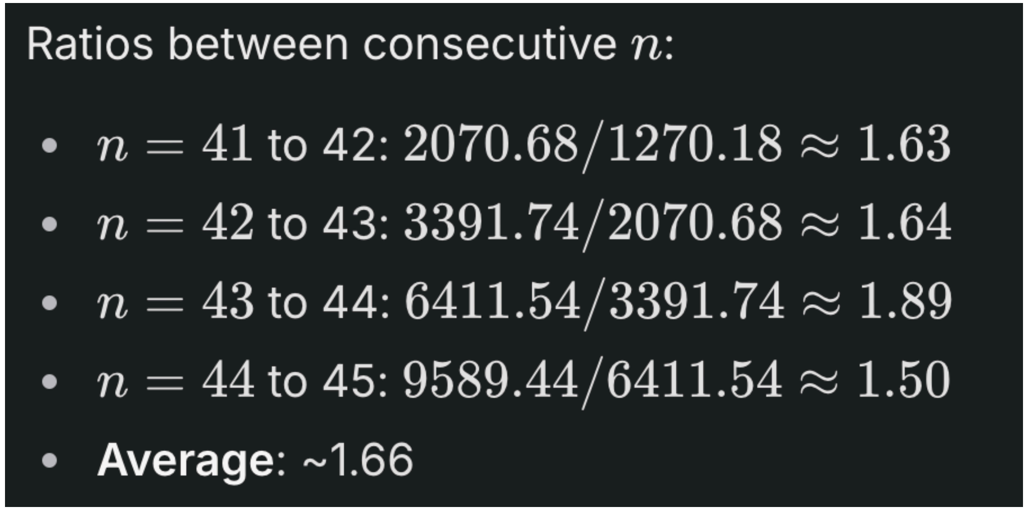
How well does this represent practice?

**Observation:** Time grows fast — roughly 1.6x per n.

**Hypothesis:** Exponential growth, like $T(n) = a * b^n$?

| n | Time (ms) |
|---|---|
| 40 | 788.09 |
| 41 | 1270.18 |
| 42 | 2070.68 |
| 43 | 3391.74 |
| 44 | 6411.54 |
| 45 | 9589.44 |
| 50 | 100329.11 |

**Tested on my machine**



Ratios between consecutive $n$:

- $n =$ 41 to 42: $2070.68/1270.18 \approx 1.63$
- $n =$ 42 to 43: $3391.74/2070.68 \approx 1.64$
- $n =$ 43 to 44: $6411.54/3391.74 \approx 1.89$
- $n =$ 44 to 45: $9589.44/6411.54 \approx 1.50$
- **Average**: ~1.66

# Confirming Exponential Growth

$T(n) = a * b^n \rightarrow \log_2(T(n)) =$

# Confirming Exponential Growth

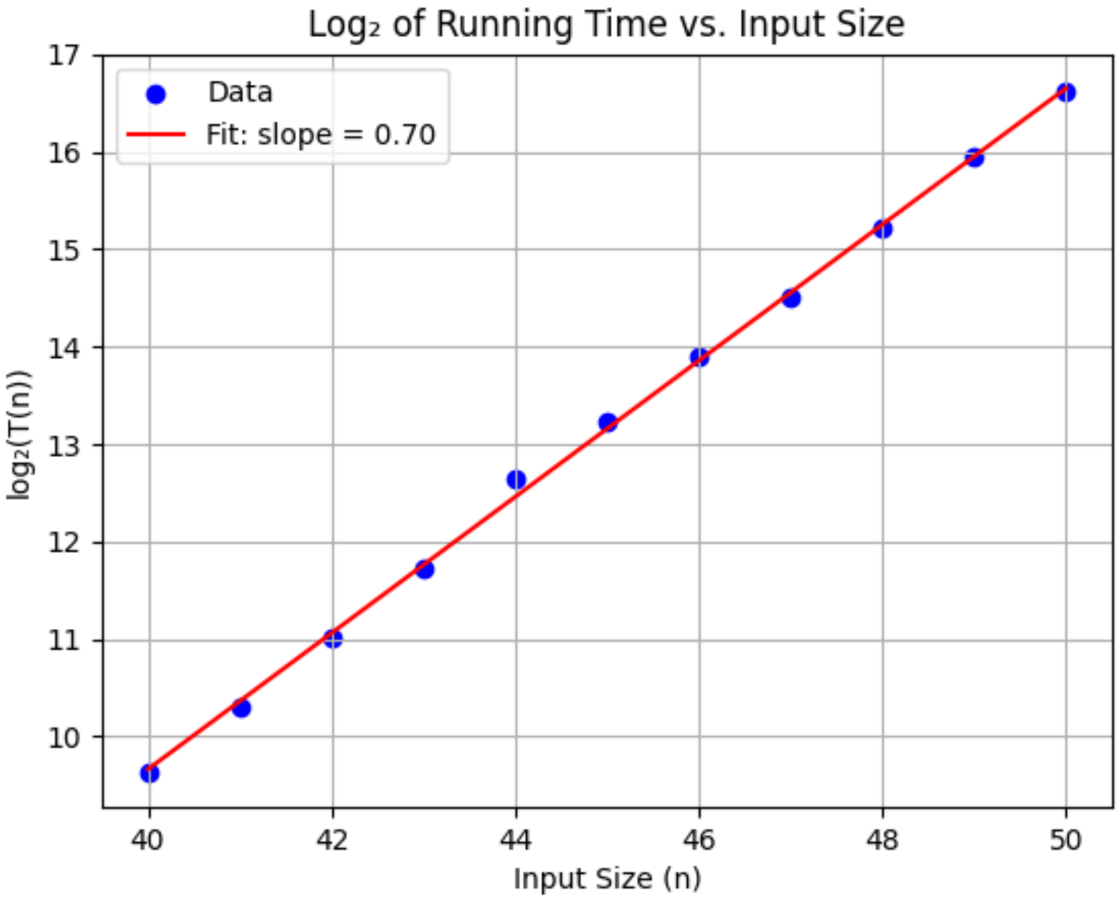
$T(n) = a * b^n \rightarrow \log_2(T(n)) = \log_2(a) + n \log_2(b)$

Calculate:

$\log_2(788.09) \approx 9.62$ (n=40)
$\log_2(100329.11) \approx 16.61$ (n=50)

Slope = $(16.61 - 9.62) / (50 - 40) \approx 0.70$

$b \approx 2^{0.7} \approx 1.62 \approx \varphi$ (1.618)
$a \approx 2^{-18.39}$



**Lab01: Do a similar empirical analysis for the 3-sum problem!!**

# Comparing predictions for T(200)

**How does our prediction for T(200) compare with Prof. Dasgupta's ($2^{92}$ s)?**

- Our empirical result: $T(n) \approx 2^{(-18.39+0.7n)}$ ms $\approx 2^{(-28.39+0.7n)}$ s
- Our prediction for $T(200) \approx 2^{111}$ s
- Dasgupta's prediction $= 2^{92}$ s
- Our predicted running time is larger by a factor of $2^{19} = 5 * 10^5$

- What can account for the difference in the results?

**Lab01: Do a similar empricial analysis for the 3-sum problem!!**

# Next time

- Abstract Data Types (OOP implementation of LinkedList)

Credits and references:

Slides based on presentations by Professors Sanjoy Das Gupta and Daniel Kane at UCSD
https://cseweb.ucsd.edu/~dasgupta/book/toc.pdf